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ON THE FORM AND POSITION OF THE SEA-LEVEL AS DEPENDENT ON SUPERFICIAL MASSES SYMMETRICALLY DISPOSED WITH RESPECT TO A RADIUS OF THE EARTH'S SURFACE.

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[continued from vol. ii, page 103.]

9. Some special values of the integrals (22)–(25), and the corresponding potentials (20) and (21), are worth deriving. These values are

a. For a point of the disturbed surface at the centre of the disturbing mass $\alpha = 0$, and (24) gives

$$I_1 = 2 \sin \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} d\gamma_1 = \pi \sin \tfrac{1}{2}\beta; \tag{26}$$

and the corresponding potential is

$$V_1 = 4 r_0 h \rho \pi \sin \tfrac{1}{2}\beta. \tag{27}$$

b. For a point at the border of the disturbing mass $\alpha = \beta$, and hence from (24) or (25)

$$\begin{aligned} I_1 = I_2 &= 2 \operatorname{arc} \sin \Big[\sin \tfrac{1}{2}\beta \sin \gamma \Big]_{\gamma = 0}^{\gamma = \frac{1}{2}\pi} \\ &= \beta, \end{aligned} \tag{28}$$

a result which is reached quite as readily from (22) or (23). The corresponding value of the potential is

$$V_1 = V_2 = 4 r_0 h \rho \beta. \tag{29}$$

c. For a point of the disturbed surface 180° from the centre of the disturbing mass $\alpha = \pi$, and (25) gives

$$I_2 = 2 \int_0^{\frac{1}{2}\pi} \frac{\sin^2 \frac{1}{2}\beta \cos^2 \gamma_2 \, d\gamma_2}{1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2}$$

$$= 2\int_0^{\frac{1}{2}\pi}\frac{(1-\cos^2\tfrac{1}{2}\beta)\,d\gamma_2}{1+\cos^2\tfrac{1}{2}\beta\tan^2\gamma_2} \tag{30}$$

$$= \pi - 2\cos\tfrac{1}{2}\beta \text{ arc tan}\Big[\tan(\cos\tfrac{1}{2}\beta\tan\gamma_2)\Big]_0^{\frac{1}{2}\pi}$$

$$= 2\pi\sin^2\tfrac{1}{4}\beta;$$

and the potential is

$$V_2 = 8r_0 h\rho\pi\sin^2\tfrac{1}{4}\beta. \tag{31}$$

d. Suppose $\beta=\pi$; in other words, let the attracting mass cover the whole sphere; then (24) gives

$$I_1 = \pi, \tag{32}$$

and we have the following well known approximate value for the potential of a spherical shell for a point on its exterior or interior surface, viz:—

$$4r_0 h\rho\pi. \tag{33}$$

This result follows also from (27) or (29), if we make $\beta=\pi$.

10. In deriving the expressions (20) and (21) for the potential of the disturbing mass, it was assumed that a sufficient degree of approximation is attained if quantities of the orders u/r_0, v/r_0, and upwards are neglected. The grounds of this assumption need to be examined with some care. For this purpose we shall derive the exact expressions, in form at least, for the potential of the disturbing mass at its centre, at its border, and at 180° from its centre. A comparison of these exact values with the approximate values given in (27), (29), and (31) will show the order of approximation of (20) and (21).

We will first write down the exact expression for the potential of a spherical shell of uniform thickness and density, for a point within its bounding surfaces. This expression will be useful as a check on formulas relating to partial shells. Let the radius of the interior surface of the complete shell be r_0, the uniform thickness h, the uniform density ρ, and the distance of the attracted point from the centre of the shell r_0+v; then the potential is *

$$V=4r_0 h\rho\pi\left[1+\frac{h}{2(r_0+v)}-\frac{v^2}{2(r_0+v)h}+\frac{hv}{2(r_0+v)r_0}-\frac{v^3}{6(r_0+v)r_0 h}\right]. \tag{34}$$

11. Let the notation be the same as that used heretofore; i. e. let r be the

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* $2\rho\pi\left[(r_0+h)^2-\frac{1}{3}(r_0+v)^2-\frac{2}{3}\frac{r_0^3}{r_0+v}\right]$.

See Price's *Calculus*, Vol. III. p. 299.

radius-vector of any point of the attracting mass, r' that of any point of the disturbed surface; ρ the density, h the thickness, and β the angular radial extent of the mass; and θ the angular distance between the attracted and attracting points, the origin being at the centre of the sphere. For points of the disturbed surface lying above the undisturbed surface, r will be less than $r' = r_0 + v$ over the range r_0 to r', and r will be greater than r' over the range r' to $r_0 + h$. Bearing these facts in mind, it follows that the exact value of the potential of the disturbing mass, for the point where its axis pierces the disturbed surface, is

$$V = 2\rho\pi \int_{r_0}^{r_0+v} r^2 dr \int_0^{\beta} \frac{\sin\theta\, d\theta}{\sqrt{(r^2 + r'^2 - 2rr'\cos\theta)}}$$

$$+\ 2\rho\pi \int_{r_0+v}^{r_0+h} r^2 dr \int_0^{\beta} \frac{\sin\theta\, d\theta}{\sqrt{(r^2 + r'^2 - 2rr'\cos\theta)}}$$

$$= 2\rho\pi \int_{r_0}^{r_0+v} \left[\sqrt{[4rr'\sin^2\tfrac{1}{2}\beta + (r'-r)^2]} - (r'-r)\right]\frac{rdr}{r'}$$

$$+\ 2\rho\pi \int_{r_0+v}^{r_0+h} \left[\sqrt{[4rr'\sin^2\tfrac{1}{2}\beta + (r-r')^2]} - (r-r')\right]\frac{rdr}{r'}.$$

By expansion, integration, and reduction this expression gives, to terms of the first order inclusive,

$$V = 4r_0 h\rho\pi \sin\tfrac{1}{2}\beta\left(1 + \frac{3h\sin\tfrac{1}{2}\beta - h}{4(r_0+v)\sin\tfrac{1}{2}\beta} + \frac{v(1-\sin\tfrac{1}{2}\beta)}{2(r_0+v)\sin\tfrac{1}{2}\beta} - \frac{v^2}{2(r_0+v)h\sin\tfrac{1}{2}\beta}\right). \tag{35}$$

This equation, if we make $\beta = \pi$, agrees with (34), as it should, to terms of the second order. It differs from (27) by terms which must be small unless h and v are very large. In one of the most important applications discussed by the writer, $h = 10\,000$ feet, $v = 3000$ feet and $\beta = 38°$. With these values, since r_0 is in round numbers 21 000 000 feet, the quantity in the brackets of (35) differs from unity by less than 1 / 10 000. If $\beta = 60°$, which is (as shown in the sequel) the angular extent of mass required to produce the maximum elevation of the disturbed surface at the centre of the mass, the quantity within the brackets of (35) exceeds unity by less than 1 / 8000, using the above value of h and v.

12. Similarly, if the attracted point be at the border of the attracting mass, the exact value of the potential is

$$V = 2\rho \int_{r_0}^{r_0+v} \frac{r}{r'}\, dr \int_0^{\frac{1}{2}\pi} \sqrt{[(r'-r)^2 + 4rr'\sin^2\tfrac{1}{2}\theta_2]}\,.\, d\lambda$$

$$+ 2\rho \int_{r_0 + v}^{r_0 + h} \frac{r}{r'} dr \int_0^{\frac{1}{2}\pi} \sqrt{[(r - r')^2 + 4rr' \sin^2 \tfrac{1}{2}\theta_2]} \,.\, d\lambda$$

$$- \rho\pi \int_{r_0}^{r_0 + v} (r' - r) \frac{r}{r'} \, dr - \rho\pi \int_{r_0 + v}^{r_0 + h} (r - r') \frac{r}{r'} \, dr,$$

in which

$$\sin^2 \tfrac{1}{2}\theta_2 = \frac{\sin^2 \beta \cos^2 \lambda}{1 - \sin^2 \beta \sin^2 \lambda}.$$

Introducing this value, the first of the above integrals relative to λ becomes

$$(r' + r) \int_0^{\frac{1}{2}\pi} \sqrt{\left\{ \frac{\sin^2 \beta \cos^2 \lambda + \left(\frac{r' - r}{r' + r} \right)^2 \cos^2 \beta}{1 - \sin^2 \beta \sin^2 \lambda} \right\}} \,.\, d\lambda.$$

Since the numerator of the element-function in this integral is greater than $\sin \beta \cos \lambda$, and less than $\sin \beta \cos \lambda + \frac{r' - r}{r' + r} \cos \beta$, the value of the integral lies between

$$(r' + r)\beta \quad \text{and} \quad (r' + r)\beta \left\{ 1 + \frac{r' - r}{r' + r} \frac{\cos \beta}{\beta} \int_0^{\frac{1}{2}\pi} \frac{d\lambda}{\sqrt{(1 - \sin^2 \beta \sin^2 \lambda)}} \right\}.$$

Therefore, if Q_1 denote a suitable factor, the exact value of the integral may be written thus: —

$$(r' + r)\beta \left\{ 1 + Q_1 \frac{r' - r}{r' + r} \right\}.$$

Likewise represent the exact value of the second integral relative to λ by

$$(r + r')\beta \left\{ 1 + Q_2 \frac{r - r'}{r + r'} \right\}.$$

Then, to terms of the first order, inclusive, the above expression for the potential becomes

$$V = 4 r_0 h \rho \beta \left\{ \begin{array}{l} 1 + \dfrac{h(3 - Q_2)}{4(r_0 + v)} - \dfrac{v(1 + Q_2)}{2(r_0 + v)} + \\ \dfrac{v^2(Q_1 + Q_2)}{4 r_0 h} - \dfrac{\pi}{\beta} \cdot \dfrac{h^2 - 2hv + 2v^2}{8(r_0 + v)h} \end{array} \right\}. \qquad (36)$$

The first term of this agrees with (29). The factors Q_1 and Q_2 lie between 0 and the elliptic of the first species

$$\frac{\cos \beta}{\beta} \int_0^{\frac{1}{2}\pi} \frac{d\lambda}{\sqrt{(1 - \sin^2 \beta \sin^2 \lambda)}}.$$

When $\beta = \frac{1}{2}\pi$, or when the attracting stratum covers a whole hemisphere, $Q_1 = Q_2 = 0$, and (36) agrees, as it should, with the half of (34) to terms of the second order; and in this case if $h = 10\,000$ feet and $v = 3000$ feet, the quantity in the parenthesis of (36) exceeds unity by 1 / 4000. When $\beta < \frac{1}{2}\pi$, Q_1 and Q_2 will exceed 0, and the degree of approximation of (36) will be somewhat greater than in the case just discussed.

13. For a point of the disturbed surface 180° from the centre of the disturbing mass, the exact value of the potential is

$$V = 2\rho\pi \int_{r_0}^{r_0+h} r^2 dr \int_0^{\beta} \frac{\sin\theta\, d\theta}{\sqrt{(r^2 + r'^2 + 2rr'\cos\theta)}}$$

$$= 2\rho\pi \int_{r_0}^{r_0+h} [r + r' - \sqrt{(r^2 + r'^2 + 2rr'\cos\beta)}]\, \frac{r}{r'}\, dr.$$

Expanding, integrating with respect to r, and reducing, there results, to terms of the first order inclusive,

$$V = 8\, r_0 h\rho\pi \sin^2 \tfrac{1}{4}\beta \left\{ 1 + \frac{3h - 2v}{4\,(r_0 + v)} \right\}. \qquad (37)$$

The first term of this agrees with (31). Using the above values of r_0 and h, and $v = -1000$ feet (v being here intrinsically negative), the factor in the brackets of (37) exceeds unity by 1 / 2600.

14. The preceding expressions for the potential of the attracting mass, namely, V as defined by equations (20) and (21), are sufficient for most of the cases they were designed to meet. They possess the obvious advantage of a compact integral form. For some purposes, however, it is desirable to have V expressed in a series of spherical harmonics, or Laplace's functions. We may thereby arrive at equations (20) and (21), by a process differing from that followed in § 7, and establish a useful harmonic development of the elliptic integrals I_1 and I_2 of equations (22) to (25).

Expressions fulfilling the present requirements may be derived from equation (10) by expanding D^{-1} in a series of ascending powers of $\frac{r}{r'}$ or $\frac{r'}{r}$. Thus from equation (9) we have

$$\frac{1}{D} = \frac{1}{r'}\left[P_0 + P_1\left(\frac{r}{r'}\right) + P_2\left(\frac{r}{r'}\right)^2 + \ldots\right], \text{ when } r < r';$$

$$\frac{1}{D} = \frac{1}{r}\left[P_0 + P_1\left(\frac{r'}{r}\right) + P_2\left(\frac{r'}{r}\right)^2 + \ldots\right], \text{ when } r > r'.$$

In these equations P_0, P_1, P_2, . . . are Laplace's co-efficients of the zero, first, second, etc. order respectively. They are functions of the angular co-ordinates θ, θ', λ, λ' only.

Taking now the centre of the sphere of reference as the origin of co-ordinates, and supposing the line from which θ and θ' are reckoned to pass through the centre of the attracting mass, we shall have for a mass of uniform thickness h, all integrations in (10) independent. For that part of the disturbed surface which lies above the undisturbed surface, $r' = r_0 + v$ will fall between the extreme values of r, which are r_0 and $r_0 + h$; and hence

$$r < r' \text{ for values between } r_0 \text{ and } r_0 + v,$$
$$r > r' \text{ for values between } r_0 + v \text{ and } r_0 + h.$$

For that part of the disturbed surface lying below the undisturbed surface, $r > r'$. In both cases the limits of θ are 0 and the angular radius β of the attracting mass; and the limits of λ are 0 and 2π. Therefore for that part of the disturbed surface lying above the undisturbed surface, equation (10) gives

$$V = \rho \int_0^\beta \sin\theta\, d\theta \int_0^{2\pi} d\lambda \int_{r_0}^{r_0+v} \left[P_0 + P_1 \left(\frac{r}{r'}\right) + P_2 \left(\frac{r}{r'}\right)^2 + \ldots \right] \frac{r^2 dr}{r'}$$
$$+ \rho \int_0^\beta \sin\theta\, d\theta \int_0^{2\pi} d\lambda \int_{r_0+v}^{r_0+h} \left[P_0 + P_1 \left(\frac{r'}{r}\right) + P_2 \left(\frac{r'}{r}\right)^2 + \ldots \right] r dr. \quad (38)$$

Likewise, for that part of the disturbed surface which lies below the undisturbed surface, equation (10) gives

$$V = \rho \int_0^\beta \sin\theta\, d\theta \int_0^{2\pi} d\lambda \int_{r_0}^{r_0+h} \left[P_0 + P_1 \left(\frac{r'}{r}\right) + P_2 \left(\frac{r'}{r}\right)^2 + \ldots \right] r dr. \quad (39)$$

For brevity, let the integrals with respect to r in (38) be denoted as folfollows: —

$$J_i = \int_{r_0}^{r_0+v} \left(\frac{r}{r'}\right)^i \frac{r^2 dr}{r'} = \frac{(r_0 + v)^{i+3} - r_0^{\,i+3}}{(i+3)(r_0 + v)^{i+1}}, \quad (40)$$

$$J'_i = \int_{r_0+v}^{r_0+h} \left(\frac{r'}{r}\right)^i r dr = \frac{(r_0 + v)^i}{i-2} \left[(r_0 + v)^{-(i-2)} - (r_0 + h)^{-(i-2)} \right]. \quad (41)$$

The ambiguous form assumed by (41) when $i = 2$ is easily shown to be equal to

$$(r_0 + v)^2 \log_e \frac{r_0 + h}{r_0 + v}.$$

Similarly, let the integrals in (39) with respect to r be denoted thus: —

$$J''_i = \frac{(r_0 + v)^i}{i - 2}\left[r_0^{-(i-2)} - (r_0 + h)^{-(i-2)}\right]. \tag{42}$$

Substituting these equivalents in (38) and (39), they become respectively

$$V = \rho \int_0^\beta \sin\theta\, d\theta \int_0^{2\pi} [(J_0 + J'_0)P_0 + (J_1 + J'_1)P_1 + \ . \ . \ . \ ]\, d\lambda, \tag{43}$$

$$V = \rho \int_0^\beta \sin\theta\, d\theta \int_0^{2\pi} [J''_0 P_0 + J''_1 P_1 + \ . \ . \ . \ ]\, d\lambda. \tag{44}$$

Now it is known from the theory of spherical harmonics * that the general value of P_i is

$P_i = f_i(\cos\theta).f_i(\cos\theta')$ + terms multiplied by cosines of multiples of $(\lambda - \lambda')$. But since

$$\int_0^{2\pi} \cos i(\lambda - \lambda')\, d\lambda = 0,$$

we have to deal only with the first term of P_i. If we write for brevity $\mu = \cos\theta$, the function of $\cos\theta$ or $\cos\theta'$ involved in this first term, is defined thus: —

$$f_i(\cos\theta) = f_i(\mu) = \frac{1}{2^i \,.\, 1 \,.\, 2 \,.\, 3 \ . \ . \ . \ i} \cdot \frac{d^i(\mu^2 - 1)^i}{d\mu^i}.$$

The following important relation exists between any three consecutive values of $f_i(\mu)$, viz.: —

$$f_i(\mu) = \frac{2i - 1}{i}\mu f_{i-1}(\mu) - \frac{i - 1}{i} f_{i-2}(\mu). \tag{45}$$

The remaining integrals in (43) and (44) are therefore of the form

$$f_i(\cos\theta') \int_0^\beta f_i(\cos\theta)\sin\theta\, d\theta \int_0^{2\pi} d\lambda = 2\pi f_i(\mu') \int_{\cos\beta}^1 f_i(\mu)\, d\mu.$$

Let

$$F_i(\beta) = \int_{\cos\beta}^1 f_i(\mu)\, d\mu. \tag{46}$$

The known value of this integral is

$$F_i(\beta) = \frac{1}{2i + 1}[f_{i-1}(\cos\beta) - f_{i+1}(\cos\beta)]. \tag{47}$$

The values of V in (43) and (44) may now be written thus, replacing θ' by

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[*] See Heine, *Handbuch der Kugelfunctionen*, or Todhunter's *Treatise on Laplace's, Lamé's, and Bessel's Functions.*

by α, θ' or α being the angular distance of the attracted point from the centre of the attracting mass:—

$$V = 2\rho\pi \sum_{i=0}^{i=\infty} (J_i + J'_i) f_i(\cos\alpha) F_i(\beta), \tag{48}$$

for points of disturbed surface above undisturbed;

$$V = 2\rho\pi \sum_{i=0}^{i=\infty} J''_i f_i(\cos\alpha) F_i(\beta), \tag{49}$$

for points of disturbed surface below undisturbed.

15. The values of V in the last two equations are exact, but their applicability to practical problems is limited by the slow convergence of the series in the second members. In most cases with which we are concerned, r_0, $r_0 + v$ and $r_0 + h$ are nearly equal, and hence the convergence depends almost wholly on the convergence of the functions of α and β.

If we neglect terms of the order h/r_0, v/r_0, and upwards, equations (48) and (49) become identical and equivalent to (20) and (21). To show these facts, we expand the second members of (40), (41), and (42), and obtain, to terms of the second order,

$$J_i + J'_i = r_0 h\left[1 - \frac{1}{2r_0}\left\{(i-1)h - 2iv + (i+\tfrac{1}{2})\frac{v^2}{h}\right\}\right] \tag{50}$$

$$J''_i = r_0 h\left[1 - \frac{(i-1)h}{2r_0} + \frac{iv}{r_0}\right]. \tag{51}$$

Hence, to terms of the first order, we have *

$$J_i + J'_i = J''_i = r_0 h,$$

and (48) and (49) become

$$V = 2r_0 h\rho\pi \sum_{i=0}^{i=\infty} f_i(\cos\alpha) F_i(\beta). \tag{52}$$

Now if $\cos\psi = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\lambda - \lambda')$,

$$2\pi \sum_{i=0}^{i=\infty} f_i(\cos\alpha) F_i(\beta) = \int_0^\beta \int_0^{2\pi} \frac{\sin\theta\, d\theta\, d\lambda}{\sqrt{(1 - 2\cos\psi + 1)}}.$$

But this integral, as shown in § 7, is equivalent to I_1 or I_2 of (22) or (23) according as α is less or greater than β. This establishes the equivalence of (52) with (20)

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* This inference from (50) and (51) does not appear to be quite satisfactory. For large values of i, $J_i + J'_i$, and J''_i are less than $r_0 h$; they are each zero for $i = \infty$. The influence of the too great factor $r_0 h$ in the higher terms of (52) is counteracted, however, by the small factors $f_i(\cos\alpha)$ $F_i(\beta)$ in those terms. The order of approximation of (52) is evident from its equivalence with (20) and (21), whose order of approximation is investigated in §§ 10–13.

and (21), and furnishes the following development in polar harmonics of the elliptic integrals in (22) to (25): —

$$\left.\begin{aligned} I_1 &= \int_0^{\alpha} \sqrt{\left\{\frac{\cos p - \cos\beta}{\cos p - \cos\alpha}\right\}} \, . \, dp = \tfrac{1}{2}\pi \sum_{i=0}^{i=\infty} f_i(\cos\alpha)\, F_i(\beta), \\ &\qquad\qquad\qquad \alpha \leqq \beta \\ I_2 &= \int_0^{\beta} \sqrt{\left\{\frac{\cos p - \cos\beta}{\cos p - \cos\alpha}\right\}} \, . \, dp = \tfrac{1}{2}\pi \sum_{i=0}^{i=\infty} f_i(\cos\alpha)\, F_i(\beta). \\ &\qquad\qquad\qquad \alpha \geqq \beta \end{aligned}\right\} \qquad (53)$$

16. In order to show the forms of the second members of equations (52) and (53), we give below the first four values of f_i (cos α) and $F_i(\beta)$. The series may be easily extended by means of the relations in (45) and (47).

$$\begin{aligned} f_0(\cos\alpha) &= 1, \\ f_1(\cos\alpha) &= \cos\alpha, \\ f_2(\cos\alpha) &= \tfrac{1}{2}(3\cos^2\alpha - 1), \\ f_3(\cos\alpha) &= \tfrac{1}{2}(5\cos^3\alpha - 3\cos\alpha); \end{aligned}$$

$$\begin{aligned} F_0(\beta) &= 1 - \cos\beta, \\ F_1(\beta) &= \tfrac{1}{2}(1 - \cos^2\beta), \\ F_2(\beta) &= \tfrac{1}{2}(\cos\beta - \cos^3\beta), \\ F_3(\beta) &= \tfrac{1}{8}(6\cos^2\beta - 5\cos^4\beta - 1). \end{aligned}$$

By means of these values equation (52) may be written thus: —

$$V = 2r_0 h\rho\pi \left\{\begin{aligned} &\;2\sin^2\tfrac{1}{2}\beta + \tfrac{1}{2}\cos\alpha\sin^2\beta \\ &+ \tfrac{1}{4}(3\cos^2\alpha - 1)\sin^2\beta\cos\beta \\ &+ \tfrac{1}{16}(5\cos^3\alpha - 3\cos\alpha)(6\cos^2\beta - 5\cos^4\beta - 1) \\ &+ \ldots \end{aligned}\right\}. \qquad (54)$$

17. In case the disturbing mass is as large as the supposed ice mass of the glacial epoch, the attraction of the re-arranged free water on itself may be appreciable. To determine the exact effect of this attraction would be a work of great difficulty even if we had the requisite information, namely, an accurate knowledge of the complicated shapes of the continents and sea-bottom. But we may determine an effect which will in general exceed the probable actual effect by supposing the whole surface of the earth covered with a film of water free to assume the proper form for equilibrium under the given forces. To fit this ideal case, formula (6) may be modified in the following manner: —

Let the potential V in (6) be replaced by $V + \varDelta V$, where $\varDelta V$ is the potential due to the re-arrangement of the water. Likewise, replace the constant V_0 by $V_0 + \Delta V_0$. Then if Δv denote the corresponding change in v, equation (6) gives

$$v + \Delta v = \frac{3}{4 r_0 \rho_m \pi} (V + \Delta V - V_0 - \Delta V_0). \qquad (55)$$

Now $v + \Delta v$ may be expressed by a series of Laplace's functions (which are in this case polar harmonics), thus: —

$$v + \Delta v = r_0 c_0 (Z_0 + Z_1 + Z_2 + \ . \ . \ . \ ),$$

in which c_0 is a constant of small numerical value; and hence, denoting the density of sea water by ρ_w, we have, as shown by Laplace*,

$$\Delta V = 4 r_0^2 c_0 \rho_w \pi \left(Z_0 + \tfrac{1}{3} Z_1 + \tfrac{1}{5} Z_2 + \ . \ . \ . \ + \frac{1}{2i+1} Z_i + \ . \ . \ . \ \right).$$

V has already been expressed in equation (54) in a series of Laplace's functions. Denoting these functions in (54) for brevity by Y_0, Y_1, Y_2, etc.,

$$V = 2 r_0 h \rho \pi (Y_0 + Y_1 + Y_2 + \ . \ . \ . \ ),$$

Substituting these values of V, ΔV, and $v + \Delta v$ in (55), there results, if we make the obviously permissible substitution,

$$\tfrac{3}{2} h \frac{\rho}{\rho_m} U_0 = \frac{3}{4 r_0 \rho_m \pi} (V_0 + \Delta V_0),$$

the following equation: —

$$\left\{ \begin{array}{ll} r_0 c_0 Z_0 \left(1 - \frac{3}{1} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} (Y_0 - U_0) \\ + r_0 c_0 Z_1 \left(1 - \frac{3}{3} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} Y_1 \\ + r_0 c_0 Z_2 \left(1 - \frac{3}{5} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} Y_2 \\ + \ . \ . \ . & \\ + r_0 c_0 Z_i \left(1 - \frac{3}{2i+1} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} Y_i \end{array} \right\} = 0.$$

According to the theory of spherical harmonics, we must have in this equation the sums of the harmonics of the same order separately equal to zero, which amounts to placing each line in the equation equal to zero. Hence we have

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*Mécanique Céleste, Book III. Chap. 2.

$$v + \Delta v = r_0 c_0 \sum_{i=0}^{i=\infty} Z_i$$

$$= \tfrac{3}{2} h \frac{\rho}{\rho_m} \left\{ \sum_{i=0}^{i=\infty} \frac{Y_i}{1 - \frac{3}{2i+1} \frac{\rho_w}{\rho_m}} - \frac{U_0}{1 - 3\frac{\rho_w}{\rho_m}} \right\}. \qquad (56)$$

This equation expresses the total effect of the disturbing mass in altering the sea-level, v being the effect which would result if the ocean were an infinitely rare fluid, and Δv being the increase over v which would result under the assumed conditions. Obviously v and Δv may be expressed separately. Thus: —

$$\Delta v = \tfrac{3}{2} h \frac{\rho}{\rho_m} \left\{ \sum_{i=0}^{i=\infty} \frac{3 Y_i}{(2i+1)\frac{\rho_m}{\rho_w} - 3} - \frac{3 U_0}{\frac{\rho_m}{\rho_w} - 3} \right\}. \qquad (57)$$

[TO BE CONTINUED.]

NOTES.

8

ON THE LAST CLAUSE OF ART. 140 OF GAUSS'S THEORIA MOTUS.

The clause mentioned is as follows: —

"Calculo examinando sequentes quoque formulae inservire possunt, quarum tamen derivationem non ita difficilem brevitatis caussa supprimimus: —

$$0 = \frac{a \sin(l'' - l')}{R} - \frac{b \sin(l'' - l)}{R'} \cdot \frac{\sin(\delta' - \sigma)}{\sin \delta'} + \frac{\sin(l' - l)}{R''},$$

$$b = \frac{R' \sin \delta'}{R'' \sin \delta''} \cdot \frac{U \cos \beta \cos \beta''}{\sin(AD' - \delta) \sin \varepsilon'};$$

ubi U exprimit quotientem $\dfrac{S}{\sin(\delta' - \sigma)} = \dfrac{T \sin(t + \gamma')}{\cos(\delta' - \sigma)}$ (art. 138. aequ. 10)."

Notwithstanding the words "non ita difficilem," the derivation of these equations is not generally readily seen by the student reading the *Theoria Motus* for the first time.

The first equation is obtained from equation III of the same article, by sub-